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LETTER TO THE EDITOR

Correspondence between Berry's phase and Lewis's phase for quadratic Hamiltonians

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Abstract. We show the existence of a relation between Berry's phase and Lewis's phase for the case of quadratic Hamiltonians.

The holonomic effect in quantum mechanics known as Berry's phase has received considerable attention recently (Berry 1984). It occurs when the time-dependent parameters of a system which evolves adiabatically in time execute a complete loop in parameter space. The wavefunction of the system then acquires, in addition to the dynamical phase $\exp(-\mathrm{i}\hbar^{-1}\int_0^T E_n(t)\,\mathrm{d}t)$, a geometrical phase factor given by

$$\gamma_n(c)=\mathrm{i}\int_0^T \mathrm{d}t\langle\Psi_n(X(t))|\mathrm{d}/\mathrm{d}t|\Psi_n(X(t))\rangle \tag{1}$$

as the parameters are slowly varied along a closed loop c in the parameter space $X(t)$ in time T. $|\Psi_n(X(t))\rangle$ are the eigenstates of the instantaneous Hamiltonian $H(X(t))$.

Berry's phase has a classical analogue as an angle shift acquired by the system when its dynamical variables are expressed in action-angle variables. This angle shift has become known as Hannay's angle (Hannay 1985, Berry 1985).

Several model systems have been chosen for calculating Berry's phase and its classical analogue. One of these systems is the generalised simple harmonic oscillator, whose Hamiltonian is given by (Berry 1985, Hannay 1985)

$$H(p,q,t)=\tfrac{1}{2}(X(t)q^2+2Y(t)qp+Z(t)p^2) \tag{2}$$

the slowly varying parameters being $X(t)$, $Y(t)$ and $Z(t)$.

In this letter we would like to show that, since (1) can be transformed to the Hamiltonian of a harmonic oscillator with time-dependent frequency, there exists a connection between Berry's phase for the system with Hamiltonian (2) and Lewis's phase for the time-dependent harmonic oscillator (Lewis and Riesenfeld 1969). Interestingly enough, the phase we shall obtain is *exact* even though the system does not evolve adiabatically in time and becomes equal to Berry's result in the adiabatic limit.

Lewis and Riesenfeld (1969) showed that for a quantal system characterised by a time-dependent Hamiltonian $H(t)$ and a Hermitian invariant $I(t)$, the general solution of the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Psi(q,t)}{\partial t}=H(t)\Psi(q,t) \tag{3}$$

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is given by

$$\Psi(q, t) = \sum_n C_n \exp(i\alpha_n(t))\Psi_n(q, t). \tag{4}$$

$\Psi_n(q, t)$ are the eigenfunctions of the invariant operator

$$I\Psi_n(q, t) = \lambda_n \Psi_n(q, t) \tag{5}$$

where the eigenvalues are time independent, the coefficients C_n are constants and the phases $\alpha_n(t)$ are obtained from the equation

$$\hbar\, d\alpha_n(t)/dt = \langle\Psi_n| i\hbar\partial/\partial t - H(t)|\Psi_n\rangle. \tag{6}$$

Using this result Lewis and Riesenfeld (1969) obtained quantal solutions for a harmonic oscillator with time-dependent frequency characterised by the Hamiltonian

$$H(t) = \tfrac{1}{2}p^2 + \tfrac{1}{2}\Omega^2(t)q^2 \tag{7}$$

and the classical equation of motion

$$\ddot{q} + \Omega^2(t)q = 0 \tag{8}$$

where the dots indicate time differentiation. The matrix elements necessary to evaluate Lewis's phase are given by (Lewis and Riesenfeld 1969)

$$\langle\Psi_n|\partial/\partial t|\Psi_n\rangle = \tfrac{1}{2}i(\rho\ddot{\rho} - \dot{\rho}^2)(n + \tfrac{1}{2}) \tag{9a}$$

$$\langle\Psi_n|H(t)|\Psi_n\rangle = \tfrac{1}{2}(\dot{\rho}^2 + \Omega^2(t)\rho^2 + 1/\rho^2)(n + \tfrac{1}{2}) \tag{9b}$$

where $\rho(t)$ is a c-number quantity satisfying the equation

$$\ddot{\rho} + \Omega^2(t)\rho = 1/\rho^3. \tag{10}$$

Substituting (9) in (6) and integrating we obtain

$$\alpha_n(t) = -(n + \tfrac{1}{2}) \int_0^t dt'/\rho^2(t'). \tag{11}$$

Our aim now is to show that, using (9a) or (11), we can obtain Berry's phase and Hannay's angle for the system with Hamiltonian (2). For this system the frequency, which can be obtained from the Hamiltonian expressed in action-angle variables, is given by

$$\omega = \partial H(I, X(t), Y(t), Z(t))/\partial I = (XZ - Y^2)^{1/2}. \tag{12}$$

From (2) one can obtain equations of motion for q and p and eliminating p we get the Newtonian equation of motion for q as

$$\ddot{q} - (\dot{Z}/Z)\dot{q} + [XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z]q = 0. \tag{13}$$

The term in $\dot{q}$ can be eliminated by introducing a new coordinate $Q(t)$ given by (Berry 1985)

$$q(t) = [Z(t)]^{1/2}Q(t). \tag{14}$$

Substituting (14) in (13) one gets

$$\ddot{Q} + \{XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z + [\tfrac{1}{2}(\ddot{Z}/Z - \dot{Z}^2/Z^2) - \tfrac{1}{4}(\dot{Z}/Z)^2]\}Q = 0 \tag{15}$$

which corresponds to the equations of motion of an oscillator with parametrically forced frequency. Berry finds Hannay's angle $\Delta\theta$ by the WKB method of quantum mechanics. We shall obtain it by means of (9a) or (11).

Comparing (8) with (15) we see that we can define $\Omega^2(t)$ as

$$\Omega^2(t) = XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z + [\tfrac{1}{4}(\ddot{Z}/Z - \dot{Z}^2/Z^2) - \tfrac{1}{4}(\dot{Z}/Z)^2]. \quad (16)$$

With this connection and employing (1) and (9*a*) we get

$$\gamma_n(C) = -\tfrac{1}{2}(n+\tfrac{1}{2}) \int_0^T (\rho\ddot{\rho} - \dot{\rho}^2)\, \mathrm{d}t \quad (17)$$

where $\rho(t)$ is the solution of (10) with $\Omega^2(t)$ given by (16). It is important to point out that (17) is *exact* even when the system does not evolve slowly in time.

In order to compare with Berry's results we shall take the adiabatic limit of our problem. In this respect we define an adiabaticity parameter ε and a 'slow time' variable τ

$$\chi \equiv \chi(\tau) \qquad \tau = \varepsilon t \quad (18)$$

in terms of which $\Omega^2(\tau)$ becomes

$$\Omega^2(\tau) = \varepsilon^{-2}\{XZ - Y^2 + \varepsilon(Z'Y - Y'Z)/Z + \varepsilon^2[\tfrac{1}{2}(Z'/Z)' - \tfrac{1}{4}(Z'/Z)^2]\} \quad (19)$$

where the primes indicate differentiation with respect to τ. It has been shown by Lewis (1968) that in the adiabatic limit (19) can be solved by a series of powers in ε with the zeroth-order term given by

$$\rho_0 = \Omega^{-1/2}(\tau). \quad (20)$$

If we substitute this expression for ρ and its time derivatives in (17) we could obtain Berry's phase in the adiabatic limit. However, it is easier to calculate Lewis's phase first and then subtract the dynamical term $-\hbar^{-1}\langle\Psi_n|H(t)|\Psi_n\rangle$. Substituting (19) and (20) in (11) we get

$$\alpha_n(\tau) = -(n+\tfrac{1}{2})\left(\frac{1}{\varepsilon}\int_0^\tau (XZ - Y^2)^{1/2}\, \mathrm{d}\tau' + \frac{1}{2}\int_0^\tau \frac{(Z'Y - Y'Z)}{Z(XZ - Y^2)^{1/2}}\, \mathrm{d}\tau' + \mathrm{O}(\varepsilon)\right). \quad (21)$$

The first term on the right-hand side is the dynamical phase and the second- and higher-order terms are associated with Berry's phase. Thus we can write Berry's phase as

$$\gamma_n(C) = -\tfrac{1}{2}(n+\tfrac{1}{2}) \int_0^T \frac{(\dot{Z}Y - \dot{Y}Z)}{Z(XZ - Y^2)^{1/2}}\, \mathrm{d}t. \quad (22)$$

Hannay's angle is obtained using Berry's correspondence principle (Berry 1985, Hannay 1985)

$$\Delta\theta = -\partial\gamma_n/\partial n \quad (23)$$

as

$$\Delta\theta = \frac{1}{2}\int_0^T \frac{(\dot{Z}Y - \dot{Y}Z)}{Z(XZ - Y^2)^{1/2}}\, \mathrm{d}t \quad (24)$$

which is the same result obtained by Berry (1985).

We have thus proved that, if a quadratic time-dependent Hamiltonian can be transformed to the form given by (7), then Lewis's phase can be used to evaluate Berry's phase and Hannay's angle. Even though we discussed a particular case it is known that Lewis's theory for time-dependent systems is general and our work shows that more work will have to be done on the connection between Berry's phase and the general theory of time-dependent constants of the motion.

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